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Linear Algebra Review Guide

This guide is meant to be a supplement to your studies – make sure to review your old exams, homework sets, in-class notes, and check with your professor to best prepare for your exam. Happy studying! -- Kevin

Kayla's Order of Topics (according to the review sheet):

<u>Pre-Exam 2</u>

I. Systems of Linear Equations
   a. Row Operations, Gaussian Elimination & Forms of Solutions (#1)
II. Matrix Operations – Adding, subtraction, scalar mult., transpose (#4)
III. Diagonal, Triangular, and Symmetric Matrices (#4)
IV. Matrix Transformations (#8)
V. Matrix Inverses (#5)
VI. Determinants (#6)
VII. Vector Spaces & Subspaces (#8)
   a. Linear Combinations and Span, Linear Independence (#3)

<u>Post-Exam 2</u>

VIII. Coordinate Vectors and Change of Basis (#9)
IX. Null, Column, and Row Space (#8)
X. Eigenvalues and Eigenvectors (#10)
XI. Diagonalization (#11)
XII. Linear Transformations, Rank-Nullity Theorem, Equivalence Theorem (#8, 9)
XIII. Kernel and Range (#7, #8)

Kevin's Order of Topics (how these worksheets are written/makes sense in my mind)

I. How to speak Linear Algebra
   a. Row Operations, Gaussian Elimination & Types of Solutions (#1)
   b. Systems of Linear Equations in Vector Form, the Matrix Equation Ax = b. (#2)
   c. Linear Independence, Linear Combinations, Basis & Span (#3)
II. Matrix & Vector Arithmetic/Geometry
   a. Addition, Subtraction, Scalar Multiplication, and Transpose (#4)
   b. Inverses of Matrices & Inverse Theorems (#5)
   c. Determinants for 2x2 and for larger matrices (#6)
III. The Heart of Linear Algebra
   a. Linear Transformations, onto and one-to-one (#7)
   b. Vector Spaces & Subspaces (#8)
   c. The Invertible Matrix Theorem, Coordinate Vectors and Change of Basis (#9)
   d. Eigenvalues & Eigenvectors (#10)
IV. Applying Linear Algebra
   a. Diagonalization (#11)

Not included? – Complex Eigenvalues, Orthogonal Projections, Inner Products, Least Squares

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Worksheet 1 - Solving Systems of Linear Equations, Gaussian Elimination & Types of Solutions

A **linear equation** is any equation that can be written in the form: $\mathbf{a_1x_1 + a_2x_2 + \ldots a_nx_n = b}$, where all a's & b can be real or complex numbers

A **linear system** is any group of one or more linear equation(s) with the same variables involved

Some examples:

$$\begin{aligned} x + y + z &= 1 \\ 2x + 3y - z &= 2 \\ x + 2y - z &= 4 \end{aligned} \quad \text{and} \quad \begin{aligned} x_1 + 2x_2 - x_4 &= 1 \\ x_2 + 2x_3 - x_4 + x_5 &= -2 \\ x_2 - 4x_3 + x_5 &= 4 \end{aligned}$$

Any system of linear equations can be written in matrix form: $\boldsymbol{\mathcal{A}\vec{x} = \vec{b}}$

For example the systems above can be written as: where the matrix A contains the coefficients of the variables

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & -1 \\ 1 & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 2 & 0 & 0 & -1 \\ 0 & 1 & 2 & -1 & 1 \\ 0 & 1 & -4 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$$

and the vector x contains the variables.

An **augmented matrix** is a shortand where the vector b and the coefficient matrix are placed side-by-side:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 3 & -1 & 2 \\ 1 & 2 & 2 & 4 \end{array}\right) \text{ and } \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 & 1 & -2 \\ 0 & 1 & -4 & 0 & 1 & 4 \end{array}\right)$$

This form is useful for solving lots of linear equations quickly.

I. Gaussian Elimination

1. Rewrite the following augmented matrices in echelon form.

(a) $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 3 & -1 & 2 \\ 1 & 2 & 2 & 4 \end{array}\right)$

(b) $\left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 1 & 1 & 0 & 5 \\ 1 & 2 & -1 & 4 \end{array}\right)$

(c) $\left(\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 & 1 & -2 \\ 0 & 1 & -4 & 0 & 1 & 4 \end{array}\right)$

***Elementary Row Operations** – in augmented matrix form, it is possible to*

-Add a multiple of one row to another row

-Switch the positions of rows

-Multiply any row by a (nonzero) constant

Performing any of these operations will not change the solutions to the system. We say that systems are "row-equivalent" when they have the same solution set. We use these properties to try to rewrite the matrix in specific forms

Row Echelon Form (REF)

(1) All nonzero rows are above any rows of all zeros.
(2) Each leading entry of a nonzero row is in a column to the right of the leading entry in the row above it.
(3) All entries in a column below a leading entry are zero.

Reduced Row Echelon Form (RREF) has two more requirements:

(4) The leading entry in each nonzero row is 1.
(5) Each leading 1 is the only nonzero entry in its column.

II. Solutions to Linear Equations

2. Solve the following solutions of equations using Gaussian elimination method and determine if the system has a unique solution, no solution, or infinitely many solutions.

(a) $\begin{aligned} x+y&=1 \\ 2x-y&=2 \end{aligned}$

(b) $\begin{aligned} x-y+z&=2 \\ x+y&=5 \\ x+2y-z&=4 \end{aligned}$

(c) $\begin{aligned} 4x-2y&=2 \\ 2x-y&=2 \end{aligned}$

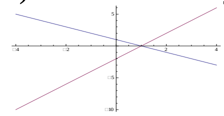
(d) $\begin{aligned} x+y+z&=1 \\ 2x+3y-z&=2 \\ x+2y-2z&=4 \end{aligned}$

(e) $\begin{aligned} 4x-2y&=4 \\ 2x-y&=2 \end{aligned}$

(f) $\begin{aligned} x+y+2z&=1 \\ 2x-y+z&=2 \\ 4x+y+5z&=4 \end{aligned}$

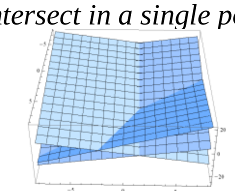
Unique Solution - If the coefficient matrix reduces to the *identity matrix* there will be a unique (numerical) solution to the system

$\begin{aligned} x+y&=1 \\ 2x-y&=2 \end{aligned}$



These two lines intersect in a single point.

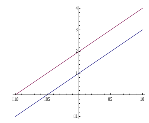
$\begin{aligned} x-y+z&=2 \\ x+y&=5 \\ x+2y-z&=4 \end{aligned}$



The solution to this system is a single point: (1,4,5.)

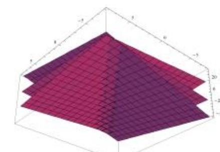
No Solution – If the system is *inconsistent* there will be no solutions. A contradiction will appear when trying to solve the system.

$\begin{aligned} 4x-2y&=2 \\ 2x-y&=2 \end{aligned}$



Here the lines are parallel & never intersect.

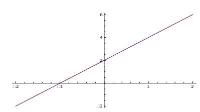
$\begin{aligned} x+y+z&=1 \\ 2x+3y-z&=2 \\ x+2y-2z&=4 \end{aligned}$



Here maybe two of the planes intersect at some line, but not all three planes will.

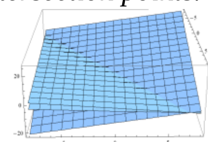
Infinitely many solutions - If, after row reduction, there are more variables than nonzero rows, the system will have a family of solutions that can be written in **parametric** form.

$\begin{aligned} 4x-2y&=4 \\ 2x-y&=2 \end{aligned}$



The two lines coincide, so they have an infinite number of intersection points.

$\begin{aligned} x+y+2z&=1 \\ 2x-y+z&=2 \\ 4x+y+5z&=4 \end{aligned}$



This system has a 1-parameter solution: it is a line in $\mathbb{R}^3$.

III. More Practice

3. Determine the values of k such that the system in unknowns x, y and z has:
   i) A unique solution
   ii) No solution
   iii) More than one solution

(a) $x + y + kz = 2$ I) $k \neq 3$ ii) *always has a solution* iii) $k = 3$

$3x + 4y + 2z = k$

$2x + 3y - z = 1$

(b) $x - 3z = -3$ i) $k \neq 2$, $k \neq -5$, $k \neq 0$ ii) $k = -5$ iii) $k = 2$, k = 0

$2x + ky - z = -2$

$x + 2y + kz = 1$

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Worksheet 2 - Systems of Linear Equations in Vector Form, the Matrix Equation, & lots of words…..

Recall: Any system of linear equations can be written in *matrix form:* $\boldsymbol{\mathcal{A}\vec{x} = \vec{b}}$

or in *vector form:* $\boldsymbol{a_1\vec{x_1} + a_2\vec{x_2} + \ \dots + a_n\vec{x_n} = \vec{b}}$

where the *vectors* in the vector form of the equation are the *columns* of the matrix A in the matrix form.

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ 2x_1 + 3x_2 - x_3 &= 0 \\ x_1 + 2x_2 - 2x_3 &= 0 \end{aligned}$$

The system on the left is a **homogenous system** because all the equations equal zero. The system on the right is **a non-homogenous** system, because not all of the equations are equal to zero.

$$\begin{aligned} x_1 - x_2 + x_3 &= 1 \\ x_1 + x_2 &= -1 \\ x_1 - x_3 &= 3 \end{aligned}$$

Consider row reduction of the augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 3 & -1 & 0 \\ 1 & 2 & -2 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The lack of a **pivot** in the 3rd column indicates a **free variable**, and an infinite number of solutions.

We can then write the solution as, letting $x_3 = t$:

$$\begin{aligned} x_1 &= -4x_3 \\ x_2 &= 3x_3 \\ x_3 &= x_3 \end{aligned} \rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \\ 1 \end{pmatrix} t$$

This solution is a 1-dimensional ***subset*** of $\mathbb{R}^3$.

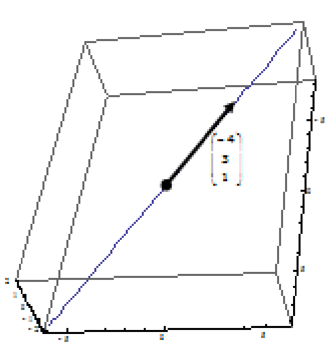
Because we got a free variable in our row reduction, we conclude that vectors $\mathbf{a_1}$, $\mathbf{a_2}$ and $\mathbf{a_3}$ are **linearly dependent**. Also, since we got 2 pivots in our reduced matrix, we can say that *these 3 vectors span a 2-dimensional subset of* $\mathbb{R}^3$ (a plane).

$$\vec{a}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}; \vec{a}_2 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}; \vec{a}_3 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}; \vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This plane will also be called the **Column Space** of matrix A. It is also the **Span** of the set **($\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_3}$).**

Because we got a free variable in our row reduction process, we have infinitely many solutions to the system. The set of all solutions form a 1-dimensional subspace of $\mathbb{R}^3$. Since this system is homogeneous, we call this solution set the **Null Space** of matrix A.

The solution was written as a vector. The Null Space consists of all multiples of this vector. Geometrically, this space is a line in $\mathbb{R}^3$, pictured below.



If that all sounded like a bunch of new words and concepts all at once, it was. We're going to build this vocabulary throughout Linear Algebra to describe properties of the system, so for now let's start to get comfortable describing solutions to systems of equations using these terms.

I. Practice

1. Solve the following systems of equations and determine if the vectors of the system are linearly independent or linearly dependent. If the vectors are linearly dependent, state the dimensions of the subset of $\mathbb{R}^n$ spanned by the vectors.

   (a) $x_1 - x_2 + x_3 = 1$

   $x_1 + x_2 = -1$

   $x_1 - x_3 = 3$

   Solution: $x_1 = 1$, $x_2 = -2$, $x_3 = -2$

$$(1)\begin{pmatrix}1\\1\\1\end{pmatrix} + (-2)\begin{pmatrix}-1\\1\\0\end{pmatrix} + (-2)\begin{pmatrix}1\\0\\-1\end{pmatrix} = \begin{pmatrix}1\\-1\\3\end{pmatrix}$$

This solution tells us the specific linear combination of $\mathbf{a}_1$ $\mathbf{a}_2$ and $\mathbf{a}_3$ that adds up to the right side vector $\mathbf{b}$.

   (b) $x_1 + 2x_2 - x_4 = 1$

   $x_2 + 2x_3 - x_4 + x_5 = -2$

   $x_2 - 4x_3 + x_5 = 4$

   Solution:

$$x_1 = -\tfrac{1}{3}x_4 + 2x_5 + 1$$
$$x_2 = \tfrac{2}{3}x_4 - 1x_5 + 0$$
$$x_3 = \tfrac{1}{6}x_4 + 0x_5 - 1$$
$$x_4 = 1x_4 + 0x_5 + 0$$
$$x_5 = 0x_4 + 1x_5 + 0$$

$$\vec{x} = \begin{bmatrix}-\frac{1}{3}\\ \frac{2}{3}\\ \frac{1}{6}\\ 1\\ 0\end{bmatrix} s + \begin{bmatrix}2\\-1\\0\\0\\1\end{bmatrix} t + \begin{bmatrix}1\\0\\-1\\0\\0\end{bmatrix}$$

There are 2 free variables, so we get a 2-dimensional subset of $\mathbb{R}^5$.

2. Give working definitions for the following terms.

(a) Consistent linear system - A linear system with at least one solution.

(b) Echelon Matrix - A rectangular matrix with three properties:

   (1) All nonzero rows are above each row of zeros.

   (2) The leading entry in each row is in a column to the right of any leading entry in a row above it.

   (3) All entries in a column below a leading entry are 0.

(c) Free Variable - A variable in a linear system that does not correspond to a pivot column.

(d) Pivot Position - A position that will contain a leading entry when the matrix is reduced to echelon form

(e) Homogeneous Equation - An equation of the form $Ax = 0$, possibly written as a system of linear equations.

(f) Nonhomogeneous equation - An equation of the form $Ax = b$ with $b \neq 0$.

(g) Null Space - (of an mxn matrix A) The set Nul(A) of all solutions to the homogeneous equation $Ax = 0$

(h) Identity Matrix - (denoted by I or I^n) A square matrix with ones of the diagonal and zeros elsewhere.

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Worksheet 3 – Linear Combinations of Vectors, Span, Dimension, & Linear Independence

A **linear combination** of a set of *vectors* (**a**$_1$, **a**$_2$, **..**, **a**$_n$) in $\mathbb{R}^m$ is a sum of multiples of those vectors:

$$\boldsymbol{a_1\overrightarrow{x_1} + a_2\overrightarrow{x_2} + \ \dots \ + a_n\overrightarrow{x_n} = \vec{b}}$$

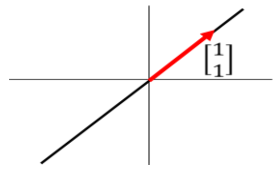
The **span** of a set of vectors is the collection of all possible of linear combinations of those vectors.

A **basis** is a *linearly independent* subset of vectors that span the entirety of the space.

I. Describing Basis & Span

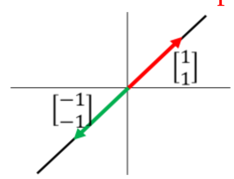
1. Describe the span of each set.

(a) $V_1 = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ $\quad span(V_1) = \left\{ x_1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \middle| x_1 \in \mathbb{R} \right\}$



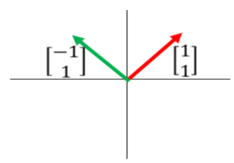
In this case there is only one vector in the set, so the span contains all the constant multiples of this vector. All of these vectors are on a line. Specifically the line y=x.

(b) $V_2 = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix} \right\}$ $\quad span(V_2) = \left\{ x_1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2 \cdot \begin{bmatrix} -1 \\ -1 \end{bmatrix} \middle| x_1, x_2 \in \mathbb{R} \right\}$



The span is the same line as before, because the new vector is just a multiple of the original, so it adds nothing new to the span. Note that either vector alone could be a **basis** for set V_2. The span of V_2 is 1-dimensional, so any vector on the line would be a basis.

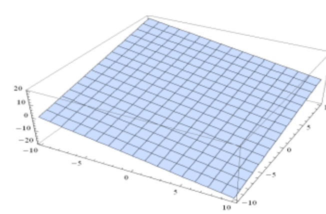
(c) $V_3 = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$ $\quad span(V_3) = \left\{ x_1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2 \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} \middle| x_1, x_2 \in \mathbb{R} \right\}$



These are not all on one line. The span is actually all of $\mathbb{R}^2$. Every vector in $\mathbb{R}^2$ can be written as a combination of the vectors in V_3, therefore Set V_3 is a basis for $\mathbb{R}^2$.

(d) $V_4 = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\}$ $\quad span(V_4) = \left\{ x_1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2 \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + x_3 \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} \middle| x_1, x_2, x_3 \in \mathbb{R} \right\}$

This time we have 3 vectors, but the new vector is a combination of the first two, so it adds nothing new to the span. This set spans all of $\mathbb{R}^2$ but is not a basis for $\mathbb{R}^{2;}$ one of the vectors is a combo of the others, so redundant. If we wrote an augmented matrix for these vectors, the RREF would have two pivot columns, and therefore the span is 2-dimensional.



(e) $V_5 = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} \right\}$ $\quad span(V_5) = \left\{ x_1 \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + x_2 \cdot \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} + x_3 \cdot \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} \middle| x_1, x_2, x_3 \in \mathbb{R} \right\}$

These are vectors from $\mathbb{R}^3$; if we formed an augmented matrix and row reduced, there are two pivots. Therefore the span is a 2-dimensional plane in $\mathbb{R}^3$. Since the span is 2-dimensional, and 2 independent vectors from V_5 will form a basis.

A set of vectors (**a**$_1$, **a**$_2$, .., **a**$_n$) is **linearly independent** if the homogenous vector equation $\boldsymbol{a_1\overrightarrow{x_1} + a_2\overrightarrow{x_2} + \ldots + a_n\overrightarrow{x_n} = \vec{0}}$ has only the **trivial solution** $x_1 = x_2 = x_n = 0$. If there is some nonzero solution, then one of the vectors can be written in terms of the others (redundancy) and the set is **linearly dependent**.

II. Determining Linear Independence

1. Are the following sets of vectors linearly independent? Describe the span of each set.

(a) $V_1 = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right\}$ V$_1$ is linearly independent and spans $\mathbb{R}^2$

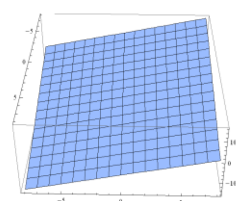
$$\left[\begin{array}{cc|c} 1 & 3 & 0 \\ 2 & 4 & 0 \end{array}\right] R_2^* = R_2 - 2R_1 \Rightarrow \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & -2 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 1 & 0 \end{array}\right] R_1^* = R_1 - 3R_2 \Rightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 = 0 \\ x_2 = 0 \end{array}$$

(b) $V_2 = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 5 \\ 6 \end{pmatrix} \right\}$ V$_2$ is linearly dependent and spans $\mathbb{R}^2$

Note that the first two vectors were already proven to be independent, and the third one is simply the sum of the first two and adds no new information to the span.

(c) $V_3 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ V$_3$ is linearly independent and spans $\mathbb{R}^3$

This set is called the STANDARD BASIS for $\mathbb{R}^3$. In general, for $\mathbb{R}^n$ the standard basis will have n vectors, each with a single 1 and zeroes elsewhere, so that they form the nxn identity matrix.



(d) $V_4 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$ V$_3$ is linearly independent and spans a 2D subset of $\mathbb{R}^3$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right] R_3^* = R_3 - R_1 \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right] R_3^* = R_3 + R_2 \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] R_1^* = R_1 - 2R_2 \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The RREF matrix has a non-pivot column, which means there is a free variable. There are 2 pivot positions, so the span of this set must be 2-dimensional (a plane in $\mathbb{R}^3$). This set is linearly dependent.

III. More Practice

2. True or false: any two vectors in $\mathbb{R}^3$ must be linearly independent.

3. True or false: any four vectors in $\mathbb{R}^3$ must span $\mathbb{R}^3$.

4. Are the vectors $v_1 = (1,0,0,0)$, $v_2 = (1,3,1,3)$, $v_3 = (1,2,1,2)$ linearly independent? If so, justify why they are. If not, write one of the vectors as a linear combination of the other two.

5. Let $v_1 = (0,1,1)$, $v_2 = (3,1,4)$, and $v_3 = (1,-1,0)$. Is v_3 in span$\{v_1,v_2\}$? If so, write v_3 as a linear combination of v_1 and v_2. If not, justify why not.

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Worksheet 4 – Matrix Arithmetic & Special Matrices

We've gotten through vocabulary; today let's think about arithmetic calculations using matrices.

A **matrix** just refers to any rectangular array. Usually the entries are just numbers, but they can be functions or operators or other things as well. We usually indicate the shape of a matrix as a pair (*m x n*) where *m = # of rows*, and *n = # of columns.*

-You can *add and subtract* matrices by adding the corresponding elements together.

-You can *multiply by a scalar* by multiplying the scalar by every element of the matrix. For both addition/subtraction & scalar multiplication the resulting matrix is the same size as the original matrix.

-You can *multiply two matrices* by taking dot products (inner products) to multiply the rows of matrix A by the columns of matrix B. The product of an (m x n) matrix and an (n x l) matrix is an (m x l) matrix.

-A **transpose** of a matrix, indicated with a superscript T, is the same matrix with the columns and rows switched. (For example a 3 x 2 matrix would transpose to a 2 x 3 matrix).

IV. Practicing Matrix Arithmetic

1. Given the following matrices A, B, C, and D, determine if the following operations can be performed.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -3 \\ 4 & 0 \\ 2 & 5 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 0 & 3 \\ 4 & 1 & -2 \end{bmatrix} \quad D = \begin{bmatrix} 3 & -2 & 1 \\ 1 & 0 & -1 \\ -3 & 2 & 1 \end{bmatrix}$$

1) 5A YES

2) A+B NO

3) C+D NO

4) A+D YES

5) D-2A YES

6) AB YES

7) BA NO

8) BC YES

9) CB YES

10) AD YES

11) $B^T D$ YES

V. Practicing Vector Arithmetic

Vectors are 1-row or 1-column matrices and therefore operate by the same rules.

• Two vectors are said to be **orthogonal** when their *scalar product* is zero.

• The **length of a vector (magnitude)** can be found by using the Pythagorean Theorem $\left\|\underset{v}{\rightarrow}\right\| = \sqrt{\underset{v}{\rightarrow} \blacksquare \underset{v}{\rightarrow}}$

• The **angle between vectors** can be found using the geometic dot product: $\underset{a}{\rightarrow} \blacksquare \underset{b}{\rightarrow} = \left\|\underset{a}{\rightarrow}\right\| \left\|\underset{b}{\rightarrow}\right\| \cos\theta$

VI. Special Matrices

1. Triangular Matrices - Given an n × n matrix A

 • A is called upper triangular if all entries below the main diagonal are 0.

 • A is called lower triangular if all entries above the main diagonal are 0.

Ex. $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ is upper triangular and $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ is lower triangular.

Note that • A matrix in REF is upper triangular.

• The transpose of an upper triangular matrix is lower triangular and vice-versa.

• The product of two Upper triangular matrices is upper triangular.

• The product of two Lower triangular matrices is lower triangular

2. Diagonal Matrices - Given an n × n matrix D

 • A matrix is called diagonal if only the diagonal entries are non-zero. If *D* is a diagonal matrix with diagonal entries $d_1, d_2, \ldots d_n$, we may write it as $\text{diag}(d_1, d_2, \ldots, d_n)$

Ex. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is diagonal.

Given two diagonal matrices $D = \text{diag}(d_1, d_2, \ldots, d_n)$ and $E = \text{diag}(e_1, e_2, \ldots, e_n)$:

• $D + E\ \text{diag}(d_1 + e_1, d_2 + e_2 \ldots, d_n + e_n)$ and $DE = \text{diag}(d_1e_1, d_2e_2 \ldots, d_ne_n)$

• For any positive integer k, $D^k = \text{diag}(d_1^k, d_2^k \ldots, d_n^k)$.

• D is invertible if and only if all the diagonal entries are non-zero and $D^{-1} = diag(\frac{1}{d_1}, \ldots, \frac{1}{d_n})$

• Diagonal matrices are both upper and lower triangular. Any matrix which is both upper and lower triangular is diagonal.

3. Symmetric Matrices - An n × n matrix A is called *symmetric* if it is equal to its transpose: $A = A^T$.
 It is called antisymmetric if it is equal to the negative of its transpose, i. e. $A = -A^T$.

Ex. $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 2 & 5 \\ 4 & 5 & 3 \end{bmatrix}$ is symmetric and

• Any diagonal matrix and its transpose are symmetric.
• If A and B are symmetric matrices, then A + B and A − B are also symmetric.
• For any scalar k, kA is also symmetric.
• A^T is symmetric since $(A^T)^T = A$, for any matrix A.

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Worksheet 5 – The Inverse of a Matrix

The **inverse of a square matrix** A is another matrix with the following properties: $A \cdot A^{-1} = A^{-1} \cdot A = I$

Here I represents the **identity matrix** of the same size as A and A^{-1}. Note that A^{-1} must be a square matrix of the same size as A. Solving the equation $\boldsymbol{AA^{-1} = I}$ is the same also solving the matrix equation $\boldsymbol{A\vec{x} = \vec{b}}$

To find the inverse, form an augmented matrix with the coefficient matrix on the left, and the identity matrix on the right. Next, row-reduce until the identity is on the left, and the inverse will be on the right.

1. Find the inverse A^{-1} of the following matrix A:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ 1 & 3 & 3 \end{bmatrix}$$

Using inverse matrices gives us a different way to solve the matrix equation, as shown here:

$$A \cdot \vec{x} = \vec{b}$$
$$A^{-1} \cdot A \cdot \vec{x} = A^{-1} \cdot \vec{b}$$
$$\vec{x} = A^{-1} \cdot \vec{b}$$

2. Find the inverse A^{-1} of the matrix A, and use it find a solution for the vector $\boldsymbol{\vec{x}}$.

$$A \cdot \vec{x} = \vec{b}$$

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 1 & 2 & -1 \end{bmatrix}; \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}; \vec{b} = \begin{bmatrix} 2 \\ 5 \\ 4 \end{bmatrix}$$

3. Solve the following systems using (a) Gaussian elimination and (b) Finding the inverse of the coefficient matrix.

(a) 3x + y = -5

2x + 3y = 6

(b) x – 2y = -8

5x + 3y = -1

(c) 5x – 2y = 1

6x + 8y = 22

(d) 2x + 3y = 4

3x + 2y = -4

(e) 3x + 2y = -17

10x + y = 0

(f) –x + 2y = 4

3x + 4y = 38

(g) 9x + 4y + 3z = -1

5x + y + 2z = 1

7x + 3y + 4z = 1

(h) 3x + 4y – 7z = -7

x – 2y + z =1

I. Inverse Matrix Theorems

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If ad-bc ≠ 0, then *A* is invertible and $A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

If ad-bc = 0, then *A* is not invertible.

(ad-bc) is called the **determinant** of *A*, written as det(A), and a 2 x 2 matrix *A* is invertible if and only if det(A) ≠ 0.

$$A \cdot \vec{x} = \vec{b}$$
$$A^{-1} \cdot A \cdot \vec{x} = A^{-1} \cdot \vec{b}$$
$$\vec{x} = A^{-1} \cdot \vec{b}$$

Let *A* be a square matrix. Then the following statements are equivalent. That is, for a given *A*, the statements are either all true or all false.

(a) *A* is an invertible matrix.

(b) *A* is row equivalent to the n x n identity matrix.

(c) *A* has *n* pivot positions.

(d) The equation Ax = 0 has only the trivial solution.

(e) The columns of *A* form a linearly independent set.

(f) The linear transformation x → Ax is one-to-one.

(g) The equation Ax = b has at least one solution for each b in R^n .

(h) The columns of *A* span R^n .

(i) The linear transformation x → Ax maps R^n onto R^n .

(j) There is an n x n matrix *C* such that CA = I.

(k) There is an n x n matrix *D* such that AD = I.

(l) A^T is an invertible matrix.

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Worksheet 6 – Determinants of Larger n x n Matrices

The **determinant** of a square matrix can be calculated in a variety of ways. It has many uses, one of which is to determine whether a matrix is invertible. For a 2x2 matrix, $\det(A) = \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc.$

For larger dimensional matrices, we need some other methods.

1. Calculate the determinant of the following matrices using all three methods.

(a) $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ 1 & 3 & 3 \end{bmatrix}$

(b) $B = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 3 & -1 \\ 4 & -4 & 0 \end{bmatrix}$

(c) $C = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 4 & 1 & 1 \\ -2 & 2 & 1 & 3 \\ 4 & 0 & 0 & -1 \end{bmatrix}$

(d) Determine if the three matrices above are *invertible*.

Laplace Cofactor Expansion - breaks a larger square matrix into several smaller pieces, until eventually you have a bunch of 2x2 determinants to evaluate. Choose a row or column to expand on. (Use one with some zeroes for ease). Alternate signs, starting with + in the upper left, or just use the formula $(-1)^{(i+j)}$

$$\text{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det(A) = +(a_{11})\begin{vmatrix} a_{ss} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - (a_{21})\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + (a_{31})\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

Row Reduction Method - Using row operations, reduce the matrix to echelon form, then the determinant is the product of the diagonal elements. Keep track of the steps in the row reduction, and back out the effects to find the original determinant.

Row Operation	**Effect on Determinant**
Add row to row	No change
Scalar mult. by k	Multiply det by k
Switch two rows	Multiply det by -1

The Basketweaving Shortcut for 3x3 matrices

Draw the first two columns to the right. Add arrows to the down-right and subtract arrows to the down-left (see diagram).

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix} =$$

$$a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - (a_{31}a_{22}a_{13} + a_{32}a_{23}a_{11} + a_{33}a_{21}a_{12})$$

This is formally called the *Sarus rule.*

Here are a few convenient rules for determinants:

$\det(AB) = \det(A)\det(B)$ $\quad$ $\det(A^T)=\det(A)$

$\det(A^{-1}) = 1/\det(A)$ $\quad$ If $\det(A)=0$, the matrix is not invertible

I. More Practice

2. If A is a 3 x 3 matrix, and det(a) = 7, what is det(2A)?

3. Let A be a 6 x 6 matrix with det(A) = 2. If the following row operations are performed to A to create a 6 x 6 matrix B, what is det(B) equal to?
    - $R_1 \rightarrow R_1 + 3R_2$
    - $R_5 \rightarrow R_6 + 2R_5$

4. If A is an invertible matrix and det(A) = 7, what is det(A^{-1})?
5. Let A be a square matrix. If det(A) = 5, what is det(A^T)?

6. Prove that if A is invertible, then det(A^{-1}) = 1/det(A).

7. Let A be a 5 x 5 matrix, and let B be obtained from A by performing the following operations in sequence:
    - Multiply the 2nd row by 3
    - Subtract 8 times the 1st row from the 4th row
    - Interchange the 2nd and 5th row
    - Add the new 5th row to the 3rd row

   Express det B in terms of det A.

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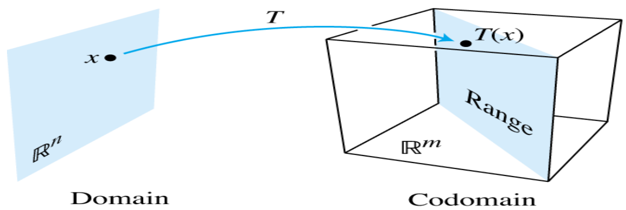
Worksheet 7 – Linear Transformations

A **transformation** (or **function** or **mapping**) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$. The set $\mathbb{R}^n$ is called the **domain** of T, and $\mathbb{R}^m$ is called the **codomain** of T.

The notation $[T: \mathbb{R}^n \rightarrow \mathbb{R}^m]$ says the domain of T is $\mathbb{R}^n$ and codomain is $\mathbb{R}^m$.

For $\mathbf{x}$ in $\mathbb{R}^n$, the vector $T(\mathbf{x})$ in $\mathbb{R}^m$ is called the **image** of $\mathbf{x}$.
The set of all images $T(\mathbf{x})$ is called the **range** of T.



Domain, codomain, and range of $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$.

In a linear transformation, for each $\mathbf{x}$ in $\mathbb{R}^n$, $T(\mathbf{x})$ is computed as $A\mathbf{x}$, where A is an mxn matrix.
For simplicity, we denote this matrix *transformation* by $[\mathbf{x} \mapsto A\mathbf{x}]$.
The *domain* of T is $\mathbb{R}^n$ when A has n columns. The *codomain* of T is $\mathbb{R}^m$ when each column of A has m entries.
Therefore, an m x n matrix transforms vectors from $\mathbb{R}^n$ into vectors from $\mathbb{R}^m$.

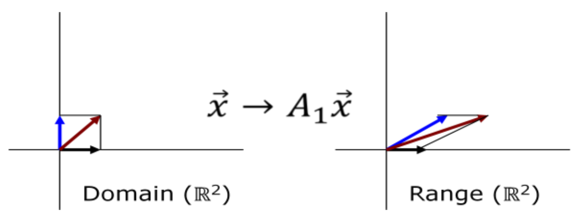
I. Applying Matrix Transformations

1. Given the matrix $A_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ show the results of applying the matrix transformation to the following vectors. State the domain and range, as well as the kind of transformation.

(a) $x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(b) $x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

(c) $x_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$



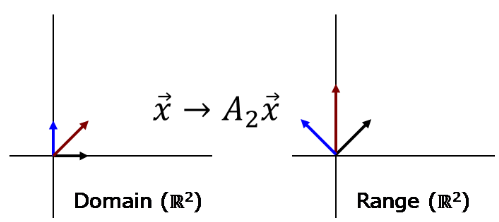
This is a shearing transformation.

2. Given the matrix $A_2 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ show the results of applying the matrix transformation to the following vectors. State the domain and range, as well as the kind of transformation.

(d) $x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

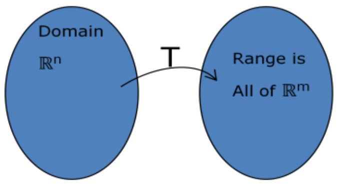
(e) $x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

(f) $x_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

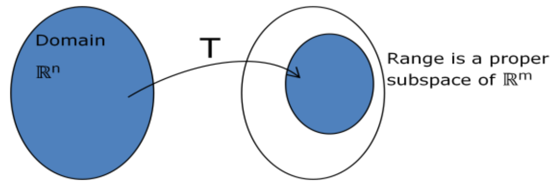


This matrix is a combination of a rotation through 45° and a stretch by a factor of √2.

A mapping T:$\mathbb{R}^n \mapsto \mathbb{R}^m$ is said to be **onto** (or surjective) if each **b** in $\mathbb{R}^m$ is the image of *at least one* **x** in $\mathbb{R}^n$. In other words the codomain of the transformation is the entirety of the range.

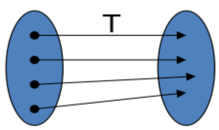


Onto

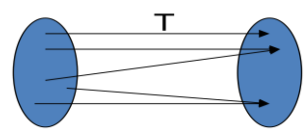


Not Onto

A mapping T:$\mathbb{R}^n \mapsto \mathbb{R}^m$ is said to be **one-to-one** (or injective) if each **b** in $\mathbb{R}^m$ is the image of *at most one* **x** in $\mathbb{R}^n$. In other words every **x** in the domain corresponds to a unique **b** in the range.



T is one-to-one

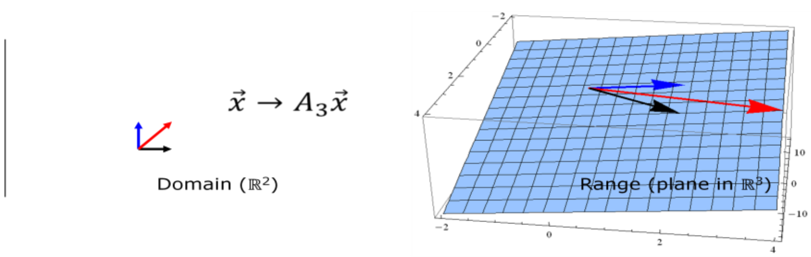


T is not one-to-one

II. Recognizing Onto and One-to-one Transformations

3. Given the matrix $A_3 = \begin{bmatrix} 1 & 0 \\ 2 & 2 \\ -1 & 3 \end{bmatrix}$ State the domain and range, as well as if the transformation is onto or one-to one.

This transformation takes a vector from $\mathbb{R}^2$ and maps it to a vector in $\mathbb{R}^3$. The range of this transformation is not the entire 3-dimensional $\mathbb{R}^3$ space. The images must be in a subset of $\mathbb{R}^3$ that has dimension (at most) 2 – a plane.



This transformation is **not onto** because the Range is not all of $\mathbb{R}^3$.

<u>A couple of quick tests to see if a transformation is one-to-one or onto:</u>

<u>More Columns than Rows?</u> – Not one-to-one

<u>More Rows than Columns?</u> –Not onto

A transformation is **onto** if the columns of A span $\mathbb{R}^m$. This happens when there is a **pivot in every row**.

A transformation is **one-to-one** iff the columns are linearly independent. This happens when there is a **pivot in every column**.

4. Given the matrix $A_4 = \begin{bmatrix} 1 & 2 & 1 & 4 & 2 \\ 0 & 0 & 1 & 3 & 5 \\ 1 & 2 & 3 & 1 & 0 \end{bmatrix}$ show the results of applying the matrix transformation to the following vectors
State the domain and range, as well as if the transformation is onto or one-to-one.

(g) $x_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

(h) $x_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

(i) $x_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and …etc

This one is a bit harder to visualize, but we are starting with vectors from $\mathbb{R}^5$, and mapping them to vectors in $\mathbb{R}^3$.The transformation is definitely not **one-to-one** because the dimension of the range (at most 3) is certainly lower than the domain (5). The transformation will be **onto** as long as the set of column vectors in the matrix spans all of $\mathbb{R}^3$. This can be checked in the usual way by row reducing the matrix and seeing that there are 3 pivot positions in the RREF form

III. How to tell that a Transformation is Linear

To be linear, a transformation must have the following properties:

Closure under addition: $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for any vectors **u** and **v** in the domain of T.

Closure under scalar multiplication: $T(c\vec{u}) = cT(\vec{u})$ for all scalars c and any vector **u** in the domain of T.

5. Are the following transformations linear?
   (a) $T(x, y) = (2x, x + y)$ Linear, Domain= $\mathbb{R}^2$, Range = $\mathbb{R}^2$, one-to-one, onto
   (b) $T(x, y) = (x - 3y, xy)$ Not Linear
   (c) $T(x, y) = (x, y, 0)$ Linear, Domain = $\mathbb{R}^3$, Codomain = 2D subspace of $\mathbb{R}^3$, not one-to-one, not onto
   (d) $T(x, y, z) = (2x, 2y, 2)$ Not linear, T(0) does not equal 0.
   (e) $T(x, y, z, w) = (2x + y, 2y + z, 2z + w)$ Linear, $\mathbb{R}^4$ to $\mathbb{R}^3$, onto, not one-to-one

For the ones that are linear, find the matrix representation (in the standard basis).

Find the dimensions of the Domain and Co-Domain, and determine if the transformation is one-to-one or onto.

IV. More Practice

6. Find the domain and codomain of the linear transformation T(x) = Ax, when $A = \begin{bmatrix} 5 & 7 & 6 & 0 \\ 1 & 0 & -2 & -2 \end{bmatrix}$

7. Pretend to redefine addition and scalar multiplication on $\mathbb{R}^2$ as follows:

$$(\vec{u} + \vec{v}) = (u_1, u_2) + (v_1, v_2) = (u_1 + v_1, 0)$$
$$k(\vec{u}) = (u_1, u_2) + (v_1, v_2) = (ku_1, 0)$$

   What vector space axioms no longer hold true?

8. Let T: $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ be multiplication by A= $\begin{bmatrix} 1 & 2 \\ -2 & -5 \end{bmatrix}$ and let e_1 and e_2 be the standard basis vectors for $\mathbb{R}^2$. Find $T(2\,e_1 + e_2)$

9. (a) Consider the subset W of $\mathbb{R}^3$ consisting of all vectors of the form (a, a + b, b). Note that (0,0,0) is in W, so W is nonempty. Show that W is a vector subspace of $\mathbb{R}^3$ (by showing that W is closed under addition and scalar multiplication).

   (b) Find two vectors v_1 and v_2 so that W = span{v_1, v_2}.

10. (a) Prove that T: $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by T(x,y) = (x+3y,x-y) is a linear transformation (by showing T satisfies the additivity and homogeneity properties).

    (b) Find the standard matrix of T.

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Worksheet 8 – Vector Spaces and Subspaces, Null Space (Kernel), and Column Space

A **vector space** is a nonempty set of vectors defined by *(1) closure under addition, (2) closure under scalar multiplication, and (3) contains the zero vector.*

A **subspace** is a vector space formed from the subset of another vector space.

I. Determining if a subset is a subspace

1. Each of the following sets is a sub***set*** of a vector space. Determine if the set is also a sub***space***.

(a) $V_1 = \{ (x, y, z)\} \epsilon\ \mathbb{R}^3 | z = 2x + 2y\}$ YES

(b) $V_2 = \{ (x, y, z)\} \epsilon\ \mathbb{R}^3 | z = 2x + 2y + 2\}$ NO – does not contain zero vector

(c) $V_3 = \{ (x, y)\} \epsilon\ \mathbb{R}^2 | z = x^2 + y^2 \leq 1\}$ NO – fails closure: (1,0) & (0,1) in space; (1,1) or (2,0) not

(d) $V_4 = \{f(t) = at^2 + bt + c\ \epsilon\ P_2 | a = b\}$ YES

(e) $V_5 = \{f(x)\ \epsilon\ C^1 | f'(x) = f(x)\}$ YES

2. Determine which of the following subsets of the vector space $\mathbb{R}^3$ are subspaces and explain.

(a) The set of S_1 vectors $(x, y, z) \epsilon\ \mathbb{R}^3$ such that $xyz = 0$. NO – fails closure (1,1,0) & (0,0,1), not (1,1,1)

(b) The set of S_2 vectors $(x, y, z) \epsilon\ \mathbb{R}^3$ such that $x + y - z = 0$. YES

(c) The set of S_3 vectors $(x, y, z) \epsilon\ \mathbb{R}^3$ such that $x + y - z = 0\ and\ 2y - 3z = 0$. YES

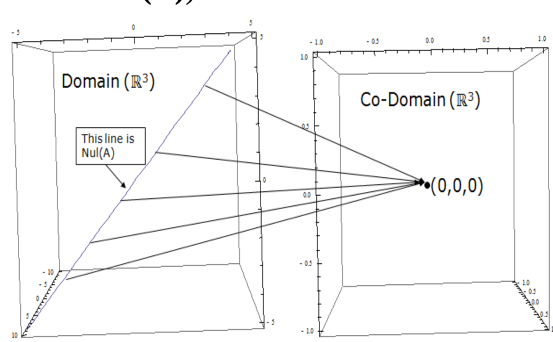
(d) The set of S_4 vectors $(x, y, z) \epsilon\ \mathbb{R}^3$ such that $x^2 - y^2 = 0$. NO – fails closure

(e) The set of S_5 vectors $(x, y, z) \epsilon\ \mathbb{R}^3$ such that $2y - 3z = 0\ and\ 2x - 3y - 1 = 0$. NO – no zero vector

(f) The set of S_6 vectors $(x, y, z) \epsilon\ \mathbb{R}^3$ such that $e^x + e^z = 0$. – NO – empty set with no zero vector

II. Types of spaces: Null Space & Column Space

The **Null Space of A** (also called the **kernel** of the linear transformation or **Nul(A)**) is the set of vectors in the domain that get mapped to the zero vector in the co-domain.



Given the following matrix A: $\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$

The null space is found as follows:

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{c} x_1 = -2x_3 \\ x_2 = x_3 \\ x_3 = x_3 \end{array} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \cdot t$$

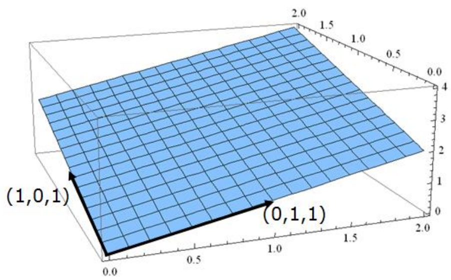
The **Column Space of A (Col(A))** is the span of the columns of A.

Given the following matrix A: $\mathbf{A}=\begin{bmatrix}1&0&2\\0&1&-1\\1&1&1\end{bmatrix}$ The column space is defined: $Col(\mathbf{A})=span\left\{\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix},\begin{bmatrix}2\\-1\\1\end{bmatrix}\right\}$

This set of vectors is *linearly dependent*. We can get the same span by leaving out the column vector that corresponds to our free variable in the row-reduced matrix (i.e. the 3rd column).

$$Col(\mathbf{A})=span\left\{\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$$

So our pared-down set only has 2 vectors, and the span of this set is a 2-dimensional subspace of $\mathbb{R}^3$ (i.e. a plane). This new set of 2 vectors is a **basis** for the column space of A.



The column space of A is this plane in $\mathbb{R}^3$ defined by the span of the basis vectors. All points in Col(A) are linear combinations of the basis vectors. (**Row space** is a similar idea just using the rows as your basis vectors instead).

1. Find the Null space and the Column space of the given matrix $\mathbf{A}=\begin{bmatrix}1&0&-2&0&3\\0&1&2&0&1\\0&0&0&1&-2\end{bmatrix}$

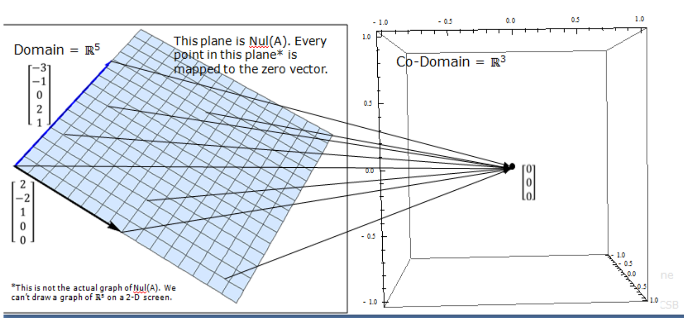
   This matrix of a linear transformation from $\mathbb{R}^5 \mapsto \mathbb{R}^3$.

We can find Nul(A) by writing down the parametric form of the solution.

There are 2 parameters (s is for x_3 and t is for x_5). Thus this is a 2-dimensional subspace of $\mathbb{R}^5$.

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}=\begin{bmatrix}2\\-2\\1\\0\\0\end{bmatrix}\cdot s+\begin{bmatrix}-3\\-1\\0\\2\\1\end{bmatrix}\cdot t$$

For the column space, we need the span of the 5 column vectors. We have a set of 5 vectors from $\mathbb{R}^3$. Columns 3 and 5 don't need to be included in the Col(A) basis since they correspond to free variables in our reduced matrix. We have 3 independent vectors in $\mathbb{R}^3$, so the span is a 3-dimensional subspace of $\mathbb{R}^3$ (i.e. **all** of $\mathbb{R}^3$).



$$Col(A)=span\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}=R^3$$

2. By the **Rank-Nullity Theorem**,

   $Rank(A)+Nullity(A)=n$, for an m x n matrix

   Or for a given transformation $T: V \rightarrow W$: $\dim(im(T))+\dim(\ker(T))=\dim(V)$

Find the **rank** *(dim(col(A))* and **nullity** *(dim(Nul(A))* of matrix A above and show that this is true.

Rank = 3, Nullity = 2, dim(domain) = 5

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Worksheet 9 – The Invertible Matrix Theorem, Coordinate Systems and Change of Basis

I. The Invertible Matrix Theorem

From last worksheet, you might be starting to realize that this entire course is really just different ways of solving the exact same problem (systems of equations) using different words, and it's confusing because every problem looks the same (and you're not wrong). Because of that though, we can go back and see that some statements logically flow from each other, given a square matrix.

Let A be an n×n matrix, and let T: $\mathbb{R}^n \to \mathbb{R}^n$ be the matrix transformation T(x)=Ax. The following statements are equivalent:

1. *A is invertible.*
2. *A has n pivots.*
3. *Nul(A)={0}.*
4. *The columns of A are linearly independent.*
5. *The columns of A span $\mathbb{R}^n$*
6. *Ax=b has a unique solution for each b in $\mathbb{R}^n$*
7. *T is invertible.*
8. *T is one-to-one.*
9. *T is onto.*

So when you're trying to do a proof of sorts, remember that these are all the same statement.

II. Coordinate Systems and Change of Basis

Recall from Precalculus: A Cartesian plane is typically defined in terms of *rectangular coordinates* (x,y), but there are times where we want to redefine that same plane in terms of *polar coordinates* (r,θ), so we convert from one set of coordinates to the other. In 3D, same idea, sometimes we have a 3D space defined by *rectangular* (x,y,z) coordinates, but we can change it to *cylindrical* or *spherical* coordinates if we so choose. We are going to generalize this idea further in terms of vector spaces, basis, and span.

Consider: The standard basis for $\mathbb{R}^2$ is $\left\{\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\right\}$. Suppose we have a different set of independent vectors from $\mathbb{R}^2$ such as $\left\{\begin{bmatrix}1\\-1\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix}\right\}$. Is this an alternate basis set for $\mathbb{R}^2$? Yes! Any set of n independent vectors form a basis for $\mathbb{R}^n$.

1. Rewrite the vector below as a linear combination of (a) the standard basis for $\mathbb{R}^2$, (b) the alternate basis for $\mathbb{R}^2$ as provided above.

$\vec{x} = \begin{bmatrix}8\\2\end{bmatrix}$

In the standard basis: $\vec{x} = \begin{bmatrix}8\\2\end{bmatrix} = 8\cdot\begin{bmatrix}1\\0\end{bmatrix} + 2\cdot\begin{bmatrix}0\\1\end{bmatrix}$

$$A\vec{c} = \vec{x}$$

$$A = \begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}; \vec{c} = \begin{bmatrix}c_1\\c_2\end{bmatrix}; \vec{x} = \begin{bmatrix}8\\2\end{bmatrix}$$

In the alternate basis: $\vec{x} = \begin{bmatrix}8\\2\end{bmatrix} = c_1\cdot\begin{bmatrix}1\\1\end{bmatrix} + c_2\cdot\begin{bmatrix}1\\-1\end{bmatrix}$

$$A\vec{c} = \vec{x} \Rightarrow A^{-1}\cdot A\vec{c} = A^{-1}\cdot\vec{x} \Rightarrow \vec{c} = A^{-1}\cdot\vec{x}$$

$$A^{-1} = \frac{1}{-2}\begin{bmatrix}-1 & -1\\-1 & 1\end{bmatrix}$$

$$[\vec{x}]_B = P_B^{-1}\vec{x}$$

$$\begin{bmatrix}8\\2\end{bmatrix} = 5\cdot\begin{bmatrix}1\\1\end{bmatrix} + 3\cdot\begin{bmatrix}1\\-1\end{bmatrix}$$

$$\vec{c} = \frac{1}{-2}\begin{bmatrix}-1 & -1\\-1 & 1\end{bmatrix}\cdot\begin{bmatrix}8\\2\end{bmatrix} = \begin{bmatrix}5\\3\end{bmatrix}$$

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Worksheet 10 – Eigenvalues & Eigenvectors

Consider a matrix transformation A that when applied to a vector returns the original vector multiplied by a scalar constant: $\boldsymbol{\mathcal{A}\vec{x} = \lambda\vec{x}}$. We will call the vector of this equation an **eigenvector** of A, and the scalar an **eigenvalue** of A. These are actually really rare in algebra, so there are important uses for them.

To solve this equation, we can rearrange to get $\boldsymbol{(\mathcal{A} - \lambda I)\vec{x} = \vec{0}}$. λ is an eigenvalue if and only if this equation has a non-trivial solution. This set of solutions to this equation is Nul$(\boldsymbol{\mathcal{A} - \lambda I})$, and this is a subspace of $\mathbb{R}^n$, called the **eigenspace**. Any vector that is in an eigenspace is mapped to another vector in that eigenspace (scaled by the eigenvalue).

I. Testing Eigenvalues & Eigenvectors

1. Consider the matrix $A = \begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}$. An eigenvalue of this matrix is λ=2. Find the associated eigenvectors.

   We need to find the nullspace of (A- λI): $A - 2I = \begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$. From this, the null space will be vectors satisfying $x_1 + 2x_2 = 0$. $\vec{x} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is an eigenvector. The eigenspace is the 1-dimesional subspace of $\mathbb{R}^2$ consisting of all multiples of $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$. We can check that the eigenvector equation is satisfied:
   $\begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \end{bmatrix} = 2\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ -- Looks good.

So now we know how to test an eigenvalue. How do we solve for eigenvalues if we don't already know them? The only way to get non-trivial solutions is if the determinant is 0. This gives us an equation we can solve for λ. A scalar λ is an eigenvalue of an n x n matrix A if and only if λ satisfies the **characteristic equation** $\boldsymbol{det(\mathcal{A} - \lambda I) = 0}$. Solving the characteristic equation gives all eigenvalues of the matrix, real and complex.

II. Solving "the Eigenvalue Problem" (Characteristic Equations)

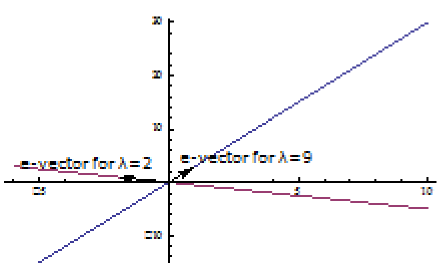
2. Consider the matrix $A = \begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}$. Find all the eigenvalues and their associated eigenspaces.

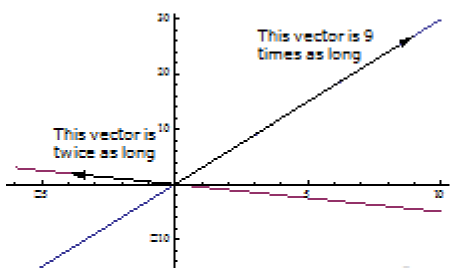
   First we solve the characteristic equations ro find the eigenvalues.

$\begin{vmatrix} 3-\lambda & 2 \\ 3 & 8-\lambda \end{vmatrix} = 0$; $(3-\lambda)(8-\lambda) - 6 = 0$; $\lambda^2 - 11\lambda + 18 = 0$; $(\lambda - 2)(\lambda - 9) = 0$; $\lambda = 2, \lambda = 9$

For $\lambda = 2$, eigenvector shown above to be $\vec{x} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. The eigenspace is $span\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\}$

For $\lambda = 9$, $A - 9I = \begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix} - \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} -6 & 2 \\ 3 & -1 \end{bmatrix}$; $3x_1 - x_2 = 0$; $\vec{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ The eigenspace is $span\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}\}$





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3. Consider the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$. Find all the eigenvalues and their associated eigenspaces.

First we solve the characteristic equations ro find the eigenvalues.

$$\det\begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0;\quad (2-\lambda)\begin{vmatrix} 3-\lambda & 1 \\ 2 & 2-\lambda \end{vmatrix} - (1)\begin{vmatrix} 2 & 1 \\ 2 & 2-\lambda \end{vmatrix} + (1)\begin{vmatrix} 2 & 1 \\ 3-\lambda & 1 \end{vmatrix} = 0$$

$$(2-\lambda)[(3-\lambda)(2-\lambda)-2] - [(2)(2-\lambda)-2] + [2-(3-\lambda)] = 0$$

$$(2-\lambda)(\lambda^2 - 5\lambda + 4) + 3\lambda - 3 = 0;\ (2-\lambda)(\lambda-1)(\lambda-4) + 3(\lambda-1) = 0$$

$$(\lambda-1)(\lambda-1)(\lambda-5) = 0;\quad \lambda = 1\ (multiplicity = 2), \lambda = 5$$

For $\lambda = 5$, $A - 5I = \begin{vmatrix} 2-5 & 2 & 1 \\ 1 & 3-5 & 1 \\ 1 & 2 & 2-5 \end{vmatrix} = \begin{vmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{vmatrix} \sim \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{vmatrix};\ \vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

The eigenspace is $span\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\}$.

For $\lambda = 1$, $A - I = \begin{vmatrix} 2-1 & 2 & 1 \\ 1 & 3-1 & 1 \\ 1 & 2 & 2-1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{vmatrix} \sim \begin{vmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix};\ \vec{x} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}\ or\ \vec{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

The eigenspace is $span\{\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\}$.

4. Consider the matrix $A = \begin{bmatrix} 5 & -4 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 5 \end{bmatrix}$. Find all the eigenvalues and their associated eigenspaces.

First we solve the characteristic equations to find the eigenvalues.

5. $\det\begin{vmatrix} 5-\lambda & -4 & 0 \\ 1 & 0-\lambda & 2 \\ 0 & 2 & 5-\lambda \end{vmatrix} = 0;\quad (5-\lambda)\begin{vmatrix} -\lambda & 2 \\ 2 & 5-\lambda \end{vmatrix} - (1)\begin{vmatrix} -4 & 0 \\ 2 & 5-\lambda \end{vmatrix} = 0$

$$(5-\lambda)[(-\lambda)(5-\lambda)-4] - [(-4)(5-\lambda)] = 0$$

$$(5-\lambda)[(-\lambda)(5-\lambda)] = 0;\ \lambda = 0\ (multiplicity\ 1);\ \lambda = 5\ (multiplicity\ 2)$$

For $\lambda = 0$, $A - 0I = \begin{vmatrix} 5 & -4 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 5 \end{vmatrix} \sim \begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & 5/2 \\ 0 & 0 & 0 \end{vmatrix};\ \begin{matrix} x_1 = -2x_3 \\ x_2 = -5/2x_3 \\ x_3 = x_3 \end{matrix}\quad \vec{x} = \begin{bmatrix} -4 \\ 5 \\ 2 \end{bmatrix}$

The eigenspace is $span\{\begin{bmatrix} -4 \\ 5 \\ 2 \end{bmatrix}\}$.

For $\lambda = 5$, $A - 5I = \begin{vmatrix} 0 & -4 & 0 \\ 1 & -5 & 2 \\ 0 & 2 & 0 \end{vmatrix} \sim \begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix};\ \begin{matrix} x_1 = -2x_3 \\ x_2 = 0 \\ x_3 = x_3 \end{matrix}\quad \vec{x} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

The eigenspace is $span\{\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}\}$.

Note that in this case. The *algebraic* multiplicity is 2 (# of factors in the char. eqn.) while the *geometric* multiplicity is 1 (dimension of the eigenspace). These don't match, so the matrix is *not diagonalizable*.

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Worksheet 11 – Diagonalization

Remember way back in Worksheet 4 how quick arithmetic is with diagonal matrices? It turns out that if an nxn matrix has n independent eigenvectors (i.e. enough to span $\mathbb{R}^n$), we will be able to perform a "similarity" transformation, to obtain a diagonal matrix that has the eigenvalues of the original matrix on the diagonal.

<u>Here is the procedure</u>: Given an n x n matrix A, find all eigenvalues and eigenvectors, then form a matrix with the eigenvectors as columns (we will call this matrix P). Next find the inverse of P. Now multiply: $P^{-1}AP=D$, D is the diagonal matrix. We can convert back by multiplying: $A=PDP^{-1}$.

1. Matrix A is given as $A = \begin{bmatrix} -4 & 2 \\ 6 & 7 \end{bmatrix}$. . Find all eigenvalues and their associated eigenvectors. Show how to use these vectors to diagonalize matrix A.

Eigenvalues:

$$\det(A-\lambda I)=0$$

$$\begin{vmatrix} -4-\lambda & 2 \\ 6 & 7-\lambda \end{vmatrix} = 0$$

$$(-4-\lambda)(7-\lambda)-(2)(6)=0$$

$$\lambda^2-3\lambda-40=0$$

$$(\lambda-8)(\lambda+5)=0$$

$$\lambda=8; \lambda=-5$$

$$\lambda_1=8 \quad \lambda_2=-5$$

$$\vec{v}_1=\begin{bmatrix}1\\6\end{bmatrix} \vec{v}_2=\begin{bmatrix}-2\\1\end{bmatrix}$$

Eigenvectors:

$$(A-8I)\vec{x}=\vec{0}$$

$$\left[\begin{array}{cc|c} -4-8 & 2 & 0 \\ 6 & 7-8 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} -12 & 2 & 0 \\ 6 & -1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 6 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$x_1=\tfrac{1}{6}x_2$$

$$eigenvector\begin{bmatrix}1\\6\end{bmatrix}$$

$$(A-(-5)I)\vec{x}=\vec{0}$$

$$\left[\begin{array}{cc|c} -4-(-5) & 2 & 0 \\ 6 & 7-(-5) & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 6 & 12 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$x_1=-2x_2$$

$$eigenvector\begin{bmatrix}-2\\1\end{bmatrix}$$

Now that we have the eigenvalues and eigenvectors, we form the matrix that has the e-vectors as columns. Call it matrix P. Also find its inverse P^{-1}. (Hint: Use the shortcut for the inverse of a 2x2 matrix).

$$P=\begin{bmatrix}1 & -2\\6 & 1\end{bmatrix} \qquad P^{-1}=\tfrac{1}{13}\begin{bmatrix}1 & 2\\-6 & 1\end{bmatrix} \qquad P^{-1}AP=\tfrac{1}{13}\begin{bmatrix}1 & 2\\-6 & 1\end{bmatrix}\begin{bmatrix}-4 & 2\\6 & 7\end{bmatrix}\begin{bmatrix}1 & -2\\6 & 1\end{bmatrix} \qquad P^{-1}AP=\tfrac{1}{13}\begin{bmatrix}1 & 2\\-6 & 1\end{bmatrix}\begin{bmatrix}8 & 10\\48 & -5\end{bmatrix}$$

Notice that matrix AP comes out to be just the eigenvectors times the eigenvalues, as expected.

$$P^{-1}AP=\tfrac{1}{13}\begin{bmatrix}1 & 2\\-6 & 1\end{bmatrix}\begin{bmatrix}8 & 10\\48 & -5\end{bmatrix}=\begin{bmatrix}8 & 0\\0 & -5\end{bmatrix}=D$$

We end up with a diagonal matrix that has the eigenvalues on the diagonal (and 0 everywhere else).

2. Matrix A is given as $A = \begin{bmatrix} 2 & 0 & -2 \\ 1 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}$. . Find all eigenvalues and their associated eigenvectors. Show how to use these vectors to diagonalize matrix A.

Eigenvalues:

$$\det(A-\lambda I)=0$$

$$\begin{vmatrix} 2-\lambda & 0 & -2 \\ 1 & 3-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{vmatrix}=0$$

$$(2-\lambda)(3-\lambda)(3-\lambda)=0$$

$$\lambda = 2(multiplicity=1)$$

$$\lambda = 3(multiplicity=2)$$

Eigenvectors:

$$(A-2I)\vec{x}=\vec{0}$$

$$\left[\begin{array}{ccc|c} 2-2 & 0 & -2 & 0 \\ 1 & 3-2 & 2 & 0 \\ 0 & 0 & 3-2 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 0 & 0 & -2 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left.\begin{array}{l} x_1 = -1x_2 \\ x_2 = 1x_2 \\ x_3 = 0 \end{array}\right\} evector = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

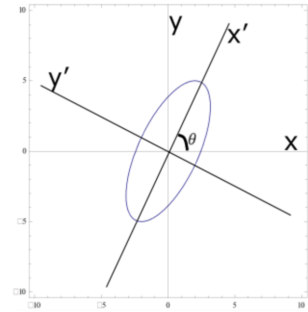
$$(A-3I)\vec{x}=\vec{0}$$

$$\left[\begin{array}{ccc|c} 2-3 & 0 & -2 & 0 \\ 1 & 3-3 & 2 & 0 \\ 0 & 0 & 3-3 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} -1 & 0 & -2 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left.\begin{array}{l} x_1 = 0x_2 - 2x_3 \\ x_2 = 1x_2 + 0x_3 \\ x_3 = 0x_2 + 1x_3 \end{array}\right\} evectors = \left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}\right\}$$

$$P = \begin{bmatrix} -1 & 0 & -2 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} -1 & 0 & -2 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \qquad P^{-1}AP = \begin{bmatrix} -1 & 0 & -2 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 & -2 \\ 1 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} -1 & 0 & -2 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = D$$

3. **The whole course in 1 Problem -**
The given equation represents an ellipse. $5x^2 - 4xy + 2y^2 = 30$



Notice that the axes of the ellipse are rotated from the standard x and y coordinate axes. Through our diagonalization process, we will find a more appropriate coordinate system where the new axes, call them x’ and y’, are aligned with the ellipse. This will simplify the equation of the ellipse. First we have to get this equation into matrix form, so we can use our linear algebra to rewrite it as:

$$5x^2 - 4xy + 2y^2 = 30 \to [x \quad y]\begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 30$$

Find eigenvalues and eigenvectors of this matrix to find the n ew axes that match up ellipse to make an alternate basis for $\mathbb{R}^{2.}$ Also find the rotation angle for the new axes.

$$\begin{vmatrix} 5-\lambda & -2 \\ -2 & 2-\lambda \end{vmatrix} = 0; \quad (5-\lambda)(2-\lambda) - 4 = 0; \quad \lambda^2 - 7\lambda + 6 = 0 \to \ (\lambda-1)(\lambda-6) = 0\,; \ \lambda = 1, \ \ \lambda = 6$$

$$\lambda = 1{:}\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \to x_1 = \frac{x_2}{2} \to \vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \lambda = 6{:}\begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix} \to x_1 = -2x_2 \to \vec{x} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Notice that these eigenvectors form an orthogonal basis for $\mathbb{R}^2$.

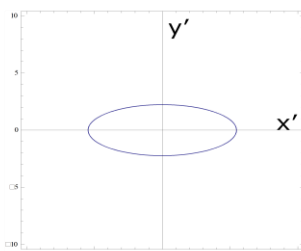
Our transformation matrix is $P = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ and its inverse is $P^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$

$$D = P^{-1}AP = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$$

Now that we have the diagonalized matrix, we can see how to write the simplified equation.

$$[x' \quad y']\begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix} = 30 \to x'^2 + 6y'^2 = 30$$

Our transformation matrix is $P = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$



The matrix that rotates the plane counterclockwise by angle θ is $\begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$

To see how this matches up with our P matrix we need to realize that matrix P does two things – a stretch and a rotation.

Each column vector of P has length $\sqrt{5}$, so we could re-write the matrix as

$P = \sqrt{5}\begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$. Now we can compare to find that $\theta = cos^{-1}\left(\frac{1}{\sqrt{5}}\right) \approx 63°$.