Name: ______Kevin Braza – KEY ___________________________ Date: __________________ IB Math A&A SL

Lesson 7.5 – Half-Angle Formulas – Equations and More Proofs

What are the half-angle formulas?

Consider the double-angle formula: $\cos(2\theta) = 1 - 2\sin^2\theta$ and $\cos(2\theta) = 2\cos^2\theta - 1$

Make the substitution $u = 2\theta$ and derive the "Half-angle formula" for sine and cosine.

$\cos\left(\frac{u}{2}\right)$

$\cos(2\theta) = 2\cos^2\theta - 1$

$\cos(u) = 2\cos^2\frac{u}{2} - 1$

$\frac{1}{2}(\cos(u) + 1) = \cos^2\frac{u}{2}$

$\cos\frac{u}{2} = \pm\sqrt{\frac{\cos u + 1}{2}}$

$\sin\left(\frac{u}{2}\right)$

$\cos(2\theta) = 1 - 2\sin^2\theta$

$\cos(u) = 1 - 2\sin^2\left(\frac{u}{2}\right)$

$\frac{-1}{2}(\cos(u) - 1) = \sin^2\frac{u}{2}$

$\sin\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}}$

1. Determine how you would prove the following identities:

$\tan\frac{u}{2} = \frac{1-\cos u}{\sin u}$ OR $\tan\frac{u}{2} = \frac{\sin u}{1+\cos u}$

$\tan\left(\frac{u}{2}\right) = \frac{\sin\left(\frac{u}{2}\right)}{\cos\left(\frac{u}{2}\right)}$

$$\tan\left(\frac{u}{2}\right) = \frac{\pm\sqrt{\frac{1-\cos u}{2}}}{\pm\sqrt{\frac{\cos u + 1}{2}}}$$

$$\tan\left(\frac{u}{2}\right) = \pm\sqrt{\frac{1-\cos u}{\cos u + 1}}$$

$\tan\left(\frac{u}{2}\right) = \pm\sqrt{\frac{1-\cos u}{1+\cos u} \times \frac{1-\cos u}{1-\cos u}}$

$\tan\left(\frac{u}{2}\right) = \pm\sqrt{\frac{(1-\cos u)^2}{1-\cos^2 u}}$

$\tan\left(\frac{u}{2}\right) = \pm\sqrt{\frac{(1-\cos u)^2}{\sin^2 u}}$

$\tan\left(\frac{u}{2}\right) = \frac{1-\cos u}{\sin u}$

OR

$\tan\left(\frac{u}{2}\right) = \pm\sqrt{\frac{1-\cos u}{1+\cos u} \times \frac{1+\cos u}{1+\cos u}}$

$\tan\left(\frac{u}{2}\right) = \pm\sqrt{\frac{1-\cos^2 u}{(1+\cos u)^2}}$

$\tan\left(\frac{u}{2}\right) = \pm\sqrt{\frac{\sin^2 u}{(1+\cos u)^2}}$

$\tan\left(\frac{u}{2}\right) = \frac{\sin u}{1+\cos u}$

Solve for all values of x in the interval $[0, 2\pi)$

2. $2 - \sin^2 x = 2\cos^2(\frac{x}{2})$

$2 - \sin^2 x = 2(\sqrt{\frac{\cos x + 1}{2}})^2$

$2 - \sin^2 x = \frac{2(\cos x + 1)}{2}$

$2 - \sin^2 x = \cos x + 1$

$2 - (1 - \cos^2 x) = \cos x + 1$

$2 - 1 + \cos^2 x = \cos x + 1$

$\cos^2 x = \cos x$

$\cos^2 x - \cos x = 0$

$\cos x\,(\cos x - 1) = 0$

$\cos x = 0;\ \ \cos x = 1$

$x = \frac{\pi}{2}, \frac{3\pi}{2} \qquad x = 0$

3. $\sin\left(\frac{x}{2}\right) + \cos x - 1 = 0$

$\pm\sqrt{\frac{1 - \cos u}{2}} + \cos x - 1 = 0$

$\pm\sqrt{\frac{1 - \cos u}{2}} = 1 - \cos x$

$\frac{1 - \cos u}{2} = (1 - \cos x)^2$

$\frac{1 - \cos u}{2} = 1 - 2\cos x + \cos^2 x$

$1 - \cos u = 2 - 4\cos x + 2\cos^2 x$

$0 = 2\cos^2 x - 3\cos x + 1$

$0 = 2\cos^2 x - 2\cos x - \cos x + 1$

$0 = 2\cos x\,(\cos x - 1) - (\cos x - 1)$

$0 = (2\cos x - 1)(\cos x - 1)$

$\cos x = \frac{1}{2};\ \ \cos x = 1$

$x = \frac{\pi}{3}, \frac{5\pi}{3} \qquad x = 0$

Use the half-angle formulas to evaluate the following. Use your calculator to verify your answers.

6. $\sin(\frac{7\pi}{8}) = \pm\sqrt{\frac{1 - \cos(\frac{7\pi}{4})}{2}} = \pm\sqrt{\frac{1 - \frac{\sqrt{2}}{2}}{2}} = \pm\sqrt{\frac{\frac{2 - \sqrt{2}}{2}}{2}} = \pm\sqrt{\frac{2 - \sqrt{2}}{4}} = \pm\frac{\sqrt{2 - \sqrt{2}}}{2} = 0.382\ radians$

7. $\cos(15°) = \pm\sqrt{\frac{1 + \cos(30°)}{2}} = \pm\sqrt{\frac{1 + \frac{\sqrt{3}}{2}}{2}} = \pm\sqrt{\frac{\frac{2 + \sqrt{3}}{2}}{2}} = \pm\sqrt{\frac{2 + \sqrt{3}}{4}} = \pm\frac{\sqrt{2 + \sqrt{3}}}{2} = 0.9659\ radians$

8. $\cos(7.5°) = \pm\sqrt{\frac{1 + \cos(15°)}{2}} = \pm\sqrt{\frac{1 + \frac{\sqrt{2 + \sqrt{3}}}{2}}{2}} = \pm\sqrt{\frac{\frac{2 + \sqrt{2 + \sqrt{3}}}{2}}{2}} = \pm\sqrt{\frac{2 + \sqrt{2 + \sqrt{3}}}{4}} = \pm\frac{\sqrt{2 + \sqrt{2 + \sqrt{3}}}}{2} = 0.9914\ radians$